Extra Questions for Keen IA Nat. Sci. Maths Students

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These difficult questions are all optional. I shall mark answers or attempts by my students, but I shan't spend very long talking about them in supervisions because they are somewhat tangential to the course. Sometimes I shall set one of them as an alternative to one of the week's standard questions: in that case it is usually better to choose the question from the lecturer's sheet unless you really feel like an extra challenge. Questions marked with an asterisk are likelier to be (even) harder than those without. Many of the questions are well-known puzzles or standard pieces of mathematics. A couple are inspired by Tripos. A few are perhaps original.

X. Assorted extension questions

X1. Take two identical cubes of side length a. Dissect the first into six congruent square-based pyramids, such that the base of each pyramid is obtained from a face of the cube, and the opposite vertex of each pyramid comes from the centre of the cube. Attach one of these pyramids, by the base, to each of the faces of the second cube.

a) Show that the resultant solid has twelve faces, each a rhombus. It is called the *rhombic dodecahedron*.

b) Find the angles within the faces.

c) Find the angle between adjacent faces of the rhombic dodecahedron.

d) Show (using the result of the last part or using the original construction) that rhombic dodecahedra can tessellate to fill space.

e) If you study materials science, relate this tessellation to the cubic close-packed structure, and to the structure of diamond.

X2. Throughout this question the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{r}$ are vectors in three dimensions; their components are real numbers. Condition E is the condition that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{r}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{r}$$

where $\mathbf{a}$ and $\mathbf{b}$ are fixed and we consider different values of $\mathbf{r}$.

a) Let $\mathbf{a} = (-101, 112, 123)$ and $\mathbf{b} = (-154, 145, 126)$. Describe fully the locus of values of $\mathbf{r}$ that satisfy condition E.

b) Let $\mathbf{a} = (-3, 4, 2)$ and $\mathbf{b} = (4, 5, -4)$. Describe fully the locus of values of $\mathbf{r}$ that satisfy condition E.

c) Show that there can be no values of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ that satisfy the equations

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{0} \quad .$$

X3. (*) In crystallography, a triclinic unit cell is a parallelepiped forming the building block for certain kinds of crystal structures. We could describe it using the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, forming three of the edges starting from an origin at one corner. Crystallographers tend to use the lengths of the edges, $a = |\mathbf{a}|$, $b = |\mathbf{b}|$, and $c = |\mathbf{c}|$, along with the angles α, β, and γ, where α is the angle between $\mathbf{b}$ and $\mathbf{c}$, β is the angle between $\mathbf{c}$ and $\mathbf{a}$, and γ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Show that the volume of the unit cell is given by

$$V = abc\sqrt{1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma} \quad .$$

X4. Take a general quadrilateral $ABCD$, and construct squares on each side of it. Represent the points A, B, C, and D with the complex numbers a, b, c, and d. Represent the centres of the squares on AB, BC, CD, and DA with the complex numbers p, q, r, and s respectively.

a) Show that

$$p = a + \frac{1}{2}(b - a)(1 + i) \quad ,$$

where p is at the centre of the square on side AB.

b) Find similar expressions for q, r, and s.

c) Find

$$\frac{p - r}{q - s} \quad .$$

d) Hence show that the lines joining the centres of 'opposite' squares have equal length, and cross at right angles.

X5. Solve the equation

$$2x^4 - 6x^3 + 9x^2 - 6x + 2 = 0 \quad .$$

(It may be useful to consider properties of x^{-1}.)

X6. A number is *the sum of two squares* if it can be written as $a^2 + b^2$, where a and b are integers. Show that, if p and q are each the sum of two squares, their product, pq, is also the sum of two squares. It may be useful to find a way to apply complex numbers.

X7. a) Let $x = \cos y$. Using the exponential form of cos, solve the equation to show that

$$y = \frac{1}{i} \log_e \left(x + i\sqrt{1 - x^2} \right) \qquad (\dagger)$$

is one solution.

b) Give a geometric interpretation of why equation (†) works. Refer to an Argand diagram.

c) Continuing your analysis from X7(a), find a formula for the other solution of $x = \cos y$ in the range $-\pi < y \leq \pi$, and show algebraically that it satisfies the expected symmetry of the cosine function. (Take $\log_e$ to give a principal value, with its imaginary part between $-\pi$ and π.)

d) By differentiating equation (†) directly, find the derivative of the equation $y = \cos^{-1} x$ in its simplest form.

e) Use equation (†) to find a function f such that

$$2\cos^{-1} x = \cos^{-1}(f(x))$$

and explain why this particular function is to be expected.

X8. (*) [Try X4 first!] Take a general triangle ABC, and construct equilateral triangles on each side of it. Let the equilateral triangle on side BC be $A'BC$. Define B' and C' similarly.

a) Represent points A, B, and C with the complex numbers a, b, and c. Work out the complex numbers corresponding to the centroids of the equilateral triangles $A'BC$, $B'AC$, and $C'AB$. Hence (or otherwise) show that the three centroids themselves form the vertices of a new equilateral triangle.

b) Join $A'A$, $B'B$, and $C'C$. Show (using complex numbers, or otherwise) that $A'A = B'B = C'C$, and that the lines cut each other at $60°$. Show further that they all meet at a point. (It may be useful to consider the circle through points A', B, and C; it would be difficult to find a formula for the point of intersection in complex numbers.) This point is called the *Torricelli point* of the triangle ABC.

c) Draw a line through A perpendicular to $A'A$, though B perpendicular to $B'B$, and through C perpendicular to $C'C$. Continue these three lines to form a large triangle. Show that this large triangle is also equilateral. (For this and the rest of the question it is likely to be easier not to use complex numbers.)

d) Show that, in any equilateral triangle, the sum of the perpendiculars from a point to the three sides is a constant, independent of the point chosen.

e) By applying this result to the large triangle, show that, of all points P in triangle ABC, the point with the least total distance to the vertices, $PA + PB + PC$, is the Torricelli Point. (Finding which point in a triangle minimizes this sum was first set as a problem by Fermat.)

X9. A parabola has the equation $y = x^2$. It is illuminated from above by rays of light parallel to the y-axis. These rays strike the parabola and are reflected. Find the gradients of the tangent and normal to the parabola at the point with x-co-ordinate x, and hence find the gradient of the reflected ray. Find the point f at which it crosses the y-axis, and show that this is independent of x. Comment on the shape of satellite dishes.

X10. Derive a series expansion for

$$\log_e \left(\frac{1+x}{1-x} \right)$$

for small values of x. By choosing a suitable value of x, give a series for $\log_e 2$. Show that the error when truncating the series after the first n (non-zero) terms is less than

$$\frac{3}{4(2n+1)9^n} \quad .$$

(This series was used to calculate tables of logarithms in the days before computers were available to do the job.)

X11. Let

$$g(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n}$$

where x is not an integer multiple of 2π. By writing $g(x)$ as the real part of a complex function, show that

$$g(x) = -\log_e \left| 2 \sin \left(\frac{x}{2} \right) \right| \quad .$$

It may be useful to consider the series expansion of the natural logarithm.

X12. The curve satisfying $x^3 + y^3 = 2xy$ is known as the *folium of Descartes*.

a) By setting up parametric equations in m, where $y = mx$, sketch the curve. Find the equation of the asymptote, the positions of any horizontal or vertical tangents to the curve, and the angle at which the curve intersects itself.

b) The area A inside the loop can be found by

$$A = \frac{1}{2} \oint x \, \mathrm{d}y - y \, \mathrm{d}x$$

where the integral is taken anticlockwise around the loop itself. Indicate on a diagram why this formula for A should work. (If you know Stokes' theorem it is more straightforward to show that it works for arbitrary loops.) Find A. It will be necessary to find the range of m corresponding to the loop, to convert the integrand into a function of m, and to express the differentials in terms of $\mathrm{d}m$.

X13. A *cycloid* is the path traced by a point on a circle, if the circle rolls without slipping along a flat surface. Take the circle to have radius a, and to roll along the x-axis such that the cycloid begins from the origin and makes arches set upon the x-axis.

a) Find parametric equations describing the cycloid. The easiest parameter to use is the angle through which the circle has turned.

b) Find the length of one arch of the cycloid in terms of the diameter of the circle.

c) Find the area underneath one arch of the cycloid in terms of the area of the circle.

d) An *intrinsic equation* is a relationship between s, the arc-length along a curve starting from the origin, and ψ, the angle between the tangent to the curve and the horizontal. (As such, $\tan\psi = \frac{\mathrm{d}y}{\mathrm{d}x}$; you should also be able to show that $\sin\psi = \frac{\mathrm{d}y}{\mathrm{d}s}$ and find a similar relation for $\cos\psi$.) Find the intrinsic equation of the cycloid.

e) (*) A piece of string is tied to the x-axis at $(2\pi a, 0)$. The cycloid is made of something solid, and the string is wrapped around it as far as the top of the first arch at $(\pi a, 2a)$. A pencil is tied to the string here. Then the string is slowly unwrapped from the cycloid, remaining under tension, while the pencil traces a curve. Show that this curve is itself the same as the original cycloid. (The cycloid is therefore known as *self-involute*.)

X14. The curve made by an idealized chain hanging between two points is known as a catenary. It satisfies the intrinsic equation $\tan\psi = ks$, where the arc-length s is measured from the lowest point of the chain. (If you study physics, derive this equation, relating k to appropriate physical parameters.) Intrinsic equations are described in X13(d). Letting the lowest point of the catenary be $(0, k^{-1})$, find its Cartesian equation.

X15. a) Factorize $x^4 + 1$ into two quadratic factors containing only real numbers. You are allowed to use complex numbers as part of the method if you like!

b) Hence, or otherwise, find

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4+1} \quad .$$

(A faster method for this kind of problem, contour integration, is taught in IB.)

X16. a) (*) Find by substitution

$$I(a,b) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(ax^2 + \frac{b}{x^2}\right)}\,\mathrm{d}x$$

where a and b are positive real numbers. The substitution

$$y = \frac{1}{2}\left(\sqrt{a}x - \frac{\sqrt{b}}{x}\right)$$

may prove useful if careful attention is paid to the limits of integration.

b) Show that the integrand in

$$J(a,b) = \int_{-\infty}^{\infty} \frac{1}{x^2} e^{-\frac{1}{2}\left(ax^2 + \frac{b}{x^2}\right)} \, \mathrm{d}x$$

tends to a definite value as x tends to zero, which means that the integral will be well-defined.

c) (*) Now find the integral $I(a,b)$ by a different method (that is, without using a substitution of something like y). Use simple substitutions to write $I(a,b)$ (separately) in terms of $I(1,ab)$, $I(ab,1)$, $I(b,a)$, and $J(b,a)$. Then by investigating $\left.\frac{\partial I(a,b)}{\partial b}\right|_a$ form and solve a differential equation for $I(a,b)$. This should give the same result as X16(a).

(These integrals appeared in paper 2 in 2008, in a Tripos question rather harder than usual.)

X17. Let

$$I = \int_0^{2\pi} \cos^{2n} x \, \mathrm{d}x \quad \text{and} \quad J = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n} x \, \mathrm{d}x$$

where n is a positive integer, possibly large.

a) By means of a diagram show that $I = 2J$.

b) Find (by any method) the first non-zero term in the Maclaurin expansion of $\log_e(\cos x)$.

c) By using

$$\cos^{2n} x = e^{2n \log_e(\cos x)}$$

with the expansion found in the last part, show that $J \approx \sqrt{\pi/n}$ for large n.

d) By writing $\cos x$ in terms of e^{ix} and e^{-ix}, show that

$$I = \frac{2\pi(2n)!}{2^{2n}(n!)^2} \quad .$$

e) Stirling's approximation (which works for large n) is sometimes given as $n! \approx n^n e^{-n}$, (which was derived in lectures) but a more accurate form is $n! \approx k n^\alpha n^n e^{-n}$, where k and α are constants which we can find. Using the results of the earlier parts together, show that they are consistent with this latter form of Stirling's approximation, and find the values of k and α required to make it work.

X18. (*) By considering the double integral

$$\int_{x=0}^{\infty} \int_{v=0}^{\lambda} e^{-ux} \cos vx \, \mathrm{d}v \, \mathrm{d}x$$

work out

$$\int_{x=0}^{\infty} \frac{\sin \lambda x}{x} \, \mathrm{d}x$$

and hence show that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x = \pi \quad .$$

X19. (*) A car sets off due east at 20mph. The driver turns the steering wheel anti-clockwise at a constant rate, such that the front wheels of the car rotate three degrees (about vertical axes) per minute. Of course this cannot continue indefinitely—the car will either tip over, lose traction, or run out of steering as its turning radius decreases—but, if it could, the car would spiral closer and closer in to a certain endpoint.

a) Sketch the path of the car. The shape is known as *Cornu's Spiral*, and has applications in optics.

b) Show that the intrinsic equation of the spiral is of the form $\psi = k\pi s^2$, and find k, given that the wheelbase of the car (the distance between the front and back wheels) is 8ft 3in. (See X13(d) for intrinsic equations.)

c) Give, in integral form, equations for the co-ordinates of the car's position in terms of the distance travelled, s.

d) By expressing the car's position as a complex number, find the resulting integral. Hence find how far away the endpoint is from where the car starts, and in which compass direction it lies.

[Make an appropriate small-angle approximation with the angle through which the front wheels have been turned in order to obtain the intrinsic equation in the form given. The exact answer would depend on which point within the car is chosen to define the speed of the body, but the error introduced by the small-angle approximation isn't as significant as the physical limitations of the car in any case.]

X20. A certain river, flowing from north to south, can be crossed using a row of n stepping-stones. Oscar is standing on one of them and cannot decide which way to go. With probability $\frac{1}{2}$ he steps one stone east (or onto the left bank if he was already on the stone furthest east). With probability $\frac{1}{2}$ he steps west in the equivalent way. Whenever he finds himself on a stepping-stone he repeats this procedure, but whenever he reaches a bank he stays there.

a) When $n = 2$, find the probability that Oscar ends up on the left bank for each of the possible starting stones.

b) Repeat the problem with $n = 3$.

c) (*) Generalize to any n.

X21. Define $S_n = \sum_{r=1}^{n} \frac{1}{r}$, the sum of the reciprocals of the first n natural numbers.

a) Show that $\lim_{n\to\infty} S_n$ is undefined.

b) The *floor function*, $\lfloor x \rfloor$, returns the largest integer which is less than or equal to x. Thus, for example, $\lfloor 7 \rfloor = 7$, $\lfloor 7.2 \rfloor = 7$, and $\lfloor 7.8 \rfloor = 7$. Sketch $y = \frac{1}{x}$ and $y = \frac{1}{\lfloor x \rfloor}$ on the same axes between $x = 1$ and $x = 7$.

c) By considering the graphs sketched above, show that the limit defining

$$\gamma = \lim_{n\to\infty} (S_n - \log_e n)$$

exists and show further that $\frac{1}{2} < \gamma < 1$. This mathematical constant γ is known as the Euler-Mascheroni constant. Show that the error in approximating S_n by $\gamma + \log_e n$ is less than $\frac{1}{2n}$, and indicate whether the approximation is larger or smaller than the true value.

d) In the game of ringboard, players repeatedly throw rubber rings at a board with hooks attached, trying to make the rings land on the hooks, and succeeding at a constant mean rate of one ring per minute. Each hook is labelled with an integer, from 1 to 13 inclusive. The players think that they can control which number they are most likely to get by *aiming*, but studies have shown that all numbers are always equally likely for all players. Lucinda is practising ringboard. There are two types of game she could play. In type I, she must first play until she gets a 1, then continue to play until she gets a 2, and so on until she reaches 13. In type II, she must also get all thirteen numbers, but they needn't be sequential: every time she gets a number that she hasn't got before, it is ticked off on a list. When all thirteen numbers have been ticked off she has finished. Find the expected length of a type I game. Using $\gamma = 0.58$ and $\log_e 13 = 2.56$ (both correct to two decimal places) work out the expected length a type II game to the nearest whole number of minutes, showing that the approximations used suffice to the required degree of accuracy.

X22. A cottage stands at the top of a cliff that runs exactly north-south. There is a lighthouse a distance b out to sea, due west of the cottage. The lighthouse randomly emits flashes of light in random directions. The beam is highly collimated (perhaps using a parabolic reflector as described in X9) so that each flash is essentially visible only at one point on the coast. The occupant of the cottage uses a long line of CCDs along the cliff top to detect the flashes, and records, every time one is detected, the northward displacement from the cottage, x.

a) Work out the probability density function for x in terms of b. Does it have a mean, median, mode, standard deviation, and interquartile range? Calculate each of these quantities that exists.

b) Work out (again for a given value of b) the probability density function that the first two flashes are received at x_1 and x_2. This is $P(x_1, x_2|b)$.

c) (*) Now look at the problem the other way round. Suppose that we've measured the first two flashes at positions x_1 and x_2. We want to use this information to work out an estimate for how far away the lighthouse is. Use Bayes' theorem. Start with a uniform prior (i.e. $P(b) = \text{const.}$, even though this cannot be normalized), as we start with no idea where the lighthouse is. Use the fact that $P(x_1, x_2)$ is independent of b to get an expression to which $P(b|x_1, x_2)$ is proportional, then normalize this to show that the probability density function for b is

$$P(b|x_1, x_2) = \frac{2(x_1 + x_2)b^2}{\pi(b^2 + x_1^2)(b^2 + x_2^2)} \quad .$$

d) (*) Does this new distribution have a mean, median, mode, standard deviation and interquartile range? Indicate how those quantities which exist would be calculated. Sketch the distribution, and work out the mode. How do you think that it is best to summarize the information that we have about the lighthouse's position?

e) (*) Suppose that more flashes are measured. What will this do to the probability distribution for b? Will it make any more quantities able to be calculated? (You needn't work out anything explicitly: the problem rapidly becomes rather complicated.)

X23. A permutation is known as a *derangement* if it leaves none of the objects in their original places. For example, there are $3! = 6$ permutations of three objects, because there are 6 ways of putting three cards, marked A, B, and C, in three envelopes, also marked A, B, and C. However, only 2 of them are derangements, because only 2 of them keep card A out of envelope A, card B out of envelope B, and card C out of envelope C. Let $f(n)$ be the number of derangements of n objects: clearly, $f(n) < n!$.

a) Find $f(n)$ for $n = 1$, 2, 3, and 4.

b) By considering where the first object can go in a derangement, and what that leaves for the rest of the objects, show that

$$f(n) = (n-1)[f(n-1) + f(n-2)] \quad .$$

Hence work out $f(5)$ and $f(6)$.

c) (*) Using the last result and mathematical induction show that

$$f(n) = nf(n-1) + (-1)^n \quad .$$

d) Using the previous result, show that

$$f(n) = n! \sum_{r=0}^{n} \frac{(-1)^r}{r!} \quad .$$

e) (*) Compare the series above with that for e^{-1}, and hence show that $f(n)$ is always the integer closest to $n!/e$.

f) Oscar has written a large number n of Christmas cards and addressed an equal number of envelopes. All of the recipients have distinct names and addresses. Oscar celebrates the completion of the writing by drinking too much mulled wine. Consequently he puts the cards into the envelopes entirely at random, though he does manage to put precisely one card in each envelope. Give the approximate probability that no-one gets the right Christmas card, and the order of the error. Also (*) work out the (exact) mean and standard deviation of the number of correctly-delivered cards. Compare this to a well-known standard distribution, and comment.

X24. In the Solar system, a small object of mass m, far enough away from the planets to be unaffected by their gravitational field, is released. Initially it is at rest a distance x_0 from the Sun. If M is the mass of the Sun,

G the gravitational constant, x the distance of the object from the Sun, and t the time since release, Newton's laws give

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{-GM}{x^2} \quad .$$

Solve this differential equation to find how long the object takes to reach a certain distance x from the Sun. It may be useful to re-cast the equation in terms of the velocity, v, and x. Hence show that the time taken to reach the centre of the Sun is

$$\pi\sqrt{\frac{x_0^3}{8GM}} \quad .$$

X25. (*) A peacock perches on a pillar at of height h. A snake emerges from a hole at the bottom of the pillar and slithers at a constant rate in a straight line along the ground away from the pillar. The ground is level. At the moment when the snake emerges, the peacock takes off. The peacock constantly alters its direction so that it is always flying towards the snake. It moves with a constant speed which is λ times that of the snake. Find the the distance from the pillar at which the peacock catches the snake. If y is height above the ground and x is horizontal displacement from the pillar, find a Cartesian equation for the path of the peacock, in the form $x = f(y)$.

X26. Differential equations with constant coefficients are susceptible to similar methods as the order of the the differential equation increases. Find the general solution in real numbers of

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = w + Ky$$

treating separately the cases $K > 0$ (setting $K = q^4$ with real q), $K = 0$, and $K < 0$ (setting $K = -4k^4$ with real k). Fourth-order differential equations can be used to describe the deflexion of beams; a choice of $K < 0$ would allow us to model the deflexion of a beam attached to lots of springs.

X27. Find the general solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = e^x + \sin 3x + e^x \cos^5 x \quad .$$

X28. Find the stationary points of the function

$$z = (3xy^2 - x^3)e^{-\frac{3}{2}(x^2+y^2)}$$

and the values of z at those points. Classify the stationary points and provide a contour sketch of the function for x and y between -2 and 2 or so.

X29. Recall the *cyclic relation* between the partial derivatives of three variables, which applies when they are related together by one constraint so that there are two degrees of freedom. Try to view the cyclic relation geometrically if possible.

a) When there are four variables related by only one constraint, we can form partial derivatives like $\frac{\partial w}{\partial x}\big|_{y,z}$. Find the value of

$$\left.\frac{\partial w}{\partial x}\right|_{y,z} \left.\frac{\partial x}{\partial y}\right|_{z,w} \left.\frac{\partial y}{\partial z}\right|_{w,x} \left.\frac{\partial z}{\partial w}\right|_{x,y} \quad .$$

b) Generalize the result above to n variables and one constraint.

c) (*) When there are four variables related by two constraints the cyclic form doesn't have a fixed numerical value. Find, nonetheless, a formula for

$$\left.\frac{\partial w}{\partial x}\right|_{y} \left.\frac{\partial x}{\partial y}\right|_{z} \left.\frac{\partial y}{\partial z}\right|_{w} \left.\frac{\partial z}{\partial w}\right|_{x}$$

in terms of only $\frac{\partial w}{\partial x}\big|_{y}$ and $\frac{\partial w}{\partial x}\big|_{z}$.

X30. In this question a, b, and c are real numbers greater than zero.

a) *Nesbitt's inequality* states that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} \quad .$$

Show using Lagrange multipliers that Nesbitt's inequality is satisfied whenever $a + b + c = 1$, and hence show that it is satisfied in general.

b) (*) Using a similar approach, but perhaps a different constraint, show that

$$\frac{50(abc)^{\frac{2}{3}}}{ab + bc + ca} + \frac{4(a + b + c)}{(abc)^{\frac{1}{3}}} \geq 27 \quad .$$

X31. a) The scalar fields ρ and p and the vector field $\mathbf{F}$ are related by $\rho\mathbf{F} = \boldsymbol{\nabla} p$. Show that $\mathbf{F}$ is perpendicular to its own curl, except perhaps in regions where $\rho = 0$.

b) The vector field $\mathbf{F}$ is solenoidal, i.e. $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$. The operator ∇^2 can be defined when acting on vectors by applying the usual

$$\nabla^2 \equiv \operatorname{div}\operatorname{grad} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

operation to each component to make a new vector. Show that

$$\operatorname{curl}\operatorname{curl}\operatorname{curl}\operatorname{curl}\mathbf{F} = \nabla^4\mathbf{F} \quad .$$

c) From the divergence theorem derive Green's theorem ("Green's second identity"),

$$\oint_{\partial V} (u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u) \cdot \mathrm{d}\mathbf{S} = \int_V u\nabla^2 v - v\nabla^2 u \,\mathrm{d}\tau$$

where ∂V denotes the closed surface bounding a region of space V, and $\mathrm{d}\tau$ is an element of volume.

X32. The velocity $\mathbf{v}$ of a fluid is a vector field with the following properties, where r is a cylindrical co-ordinate.

- The components of $\mathbf{v}$ in terms of the cylindrical polar basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ depend only on r.
- $\mathbf{v}$ is everywhere perpendicular to $\mathbf{e}_z$.
- The components of $\mathbf{v}$ are continuous in all directions everywhere.
- $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ everywhere.
- $\boldsymbol{\nabla} \times \mathbf{v} = 2\boldsymbol{\omega}$ for $r \leq R$ but $\boldsymbol{\nabla} \times \mathbf{v} = 0$ for $r > R$.

By using appropriate vector calculus theorems which exploit the symmetries, find $\mathbf{v}$ in terms of $|\boldsymbol{\omega}|$ and R. This flow pattern is known as a *Rankine vortex*; it is sometimes a reasonable approximation to the vortex motion of a fluid with a fairly low viscosity.

X33. Consider the Fourier series

$$f(x) = \frac{1}{2} + \frac{1}{2}\cos(Nx) + \sum_{n=1}^{N-1} \cos(nx) \quad .$$

a) First sketch $f(x)$ for $N = 2$ between 0 and 4π. (It is not necessary in this question to find all of the stationary points.)

b) Now sketch $f(x)$ for $N = 3$ over the same range. If you have a computer to hand, look at similar plots with larger N.

c) Now try to understand the behaviour of the series algebraically. Show that

$$f(x) = \frac{1}{2}\cot\left(\frac{x}{2}\right)\sin(Nx) \quad .$$

Explain what happens to $f(x)$ for large N (i) when x is not close to a multiple of 2π and (ii) when x is close to a multiple of 2π, indicating the behaviour on a rough sketch.

d) Now try to understand the behaviour of the series geometrically. Show the numbers $\frac{1}{2}$, e^{ix}, e^{2ix} &c. on an Argand diagram for a small value of x and a random larger value of x. Arrange the vectors nose-to-tail to form the complex sum, and relate the real part of this complex number to the behaviour observed in the previous part. (For physicists, relate also to multi-slit diffraction, and for materials scientists, relate to Bragg reflexion.)

e) What must the integral of $f(x)$ be over a complete period? (Only one term in the series contributes.)

f) (*) In the limit as $N \to \infty$, what do you expect the integral of $f(x)$ to be over any range that does not include a multiple of 2π?

g) (*) Discuss the form of the function itself in the limit as $N \to \infty$, and explain how the coefficients of the Fourier series would have been obtained from this function. We shall have to generalize the notion of a function here, by considering its effect on integrals, rather than its values, which are all undefined in the limit.

X34. This question is about Vandermonde determinants, which have applications in polynomial interpolation.

a) Show that the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

is equal to $(y - x)(z - x)(z - y)$.

b) Find

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^3 & x^3 & y^3 & z^3 \end{vmatrix}$$

in a similar form.

c) (*) Generalize this result to n variables, raised to powers $0, 1 \ldots n-1$ in an n by n determinant.

X35. The *Fibonacci numbers* F_n are defined such that $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$ for $n > 2$. Define the vector $\mathbf{v}_n = (F_n, F_{n-1})$. Find a matrix $\mathbf{M}$ such that $\mathbf{v}_{n+1} = \mathbf{M}\mathbf{v}_n$ for $n > 1$. By writing this matrix in diagonalized form, find a simpler form for powers of $\mathbf{M}$, hence find an explicit (non-recursive) formula for F_n. Use this formula to evaluate

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

which is known as the *golden ratio.*

X36. The real matrices $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ obey

$$\mathbf{X}^2 = \mathbf{Y}^2 = \mathbf{Z}^2 = \mathbf{XYZ} = -\mathbf{I}$$

where $\mathbf{I}$ is the n by n identity matrix. This question is about the representation of a four-dimensional analogue of complex numbers, known as *quaternions*, by matrices, but you do not need to know anything about quaternions to do the question.

a) Show that $\mathbf{X}$ cannot be a symmetric matrix.

b) Show that $\mathbf{XY} + \mathbf{YX} = \mathbf{0}$.

c) For the rest of the question, add the further requirement that $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are orthogonal matrices. Show that this implies that they are all antisymmetric.

d) We consider matrices of the form $\mathbf{Q}$, where

$$\mathbf{Q} = a\mathbf{I} + b\mathbf{X} + c\mathbf{Y} + d\mathbf{Z}$$

with a, b, c, and d any real numbers. The matrices are fixed constants. Thus a matrix is in the form $\mathbf{Q}$ if it is a linear combination of the four matrices above. Show that the sum, difference, and product of a pair of matrices of form $\mathbf{Q}$ is another matrix of the form $\mathbf{Q}$.

e) (*) Find the determinant of $\mathbf{Q}$ in terms of a, b, c, and d (and also n). Hence show that every non-zero matrix of the form $\mathbf{Q}$ has an inverse which is also a matrix of the form $\mathbf{Q}$, and find a formula for this inverse.

f) Find a possible set of matrices $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ that satisfy the properties. The simplest representation has $n = 4$; the elements are all 1, 0, and -1.

g) (*) Continuing the analysis of the determinants above, show, by analogy with X6, that if two integers can each be written as the sum of four squares, then their product can also be written as the sum of four squares. (In fact all integers can be written as the sum of four squares; this is one step in a proof.)

X37. (*) The scalar field $\psi = \psi(x, y)$ obeys

$$\nabla^2\psi + 10\frac{\partial\psi}{\partial x} + 34\psi = 0$$

subject to $\psi(0, y) = 0$ for all y, $\psi(x, 0) = \psi(x, \pi) = 0$ for all x, and $\psi(x, y) = \psi(x, \pi - y)$ for all x and y. In addition,

$$\lim_{x\to\infty} e^x\psi\left(x, \frac{\pi}{2}\right) = 1 \quad ,$$

$$\psi\left(\frac{\pi}{\sqrt{2}}, \frac{2\pi}{5}\right) = \frac{\pi}{\sqrt{2}}\sin\left(\frac{\pi}{5}\right) \quad \text{, and}$$

$$\psi\left(\frac{\pi}{\sqrt{32}}, \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}e^{-9\pi/\sqrt{32}} \quad .$$

Find $\psi(x, y)$.

X38. When a rope under tension is wrapped around a body without friction, it follows a *geodesic* curve on the surface. Broadly, this means that between any two points on the path of the rope, the path that it does take between those two points is the shortest that it could. Now the tip of a certain mountain is a perfect cone covered in perfectly slippery ice. Lucinda tries to climb the mountain by throwing a fixed-loop rope lasso around the summit and hauling herself up on it. What is the maximum vertex angle of the cone such that this approach could work? Note that the geodesic property holds for every part of the loop that does not include the attachment to Lucinda. This problem can be tackled with the calculus of variations, taught in IB, but doesn't really need calculus at all!

X39. Reflexions of objects in mirrors appear to have their left and right halves swapped round. Why, given the isotropy of space, is this the case when the top and bottom halves aren't swapped?