Post-Christmas IA NST Mathematics Test

January 2024

You have one hour. Complete any **two** questions from those printed below. The starred question requires knowledge of B-course material. The number of marks available for each part of a question is shown in square brackets. Begin your answer to each question on a new page, writing the question number clearly at the top. Do not submit answers to more than two questions. Calculators may not be used for this test.

Question 1

The complex numbers z and w obey $z^{12} = -1$ and $w^{12} = (2 + \sqrt{3})^6$. In this question, take the argument of a complex number to lie between $-\pi$ and π.

a) Find the possible values of z and of w, giving a general formula for each in modulus-argument form, and mark all of the points corresponding to each z and w on a single Argand diagram. [5]

b) On your diagram from (a), join each of the points z with a line to the origin. Also join with lines each of the points z to both of the *two* points w that lie closest to it in each case. [1]

c) The quadrilateral R has vertices at the origin, at those two points z that have smallest $|\arg(z)|$, and at the point w that has smallest $|\arg(w)|$. Find the lengths of all four sides of quadrilateral R, in their simplest forms. [4]

d) Find the vector area of quadrilateral R, where the number 1 on the Argand diagram corresponds to $(1, 0, 0)$ and the number i corresponds to $(0, 1, 0)$. [3]

e) Find the area of the projection of quadrilateral R onto a plane that passes through the points $(5, 0, 0)$, $(0, 10, 0)$, and $(0, 0, 15)$. [6]

f) Find the projected area of the entire shape drawn in (b) onto the plane of part (e). [1]

Question 2

a) Sketch a graph of $y = \operatorname{sech}(x)$, for both positive and negative x, finding and marking on any points of inflexion. [5]

b) Find

$$\int_{-\infty}^{\infty} \operatorname{sech}^n(x)\,\mathrm{d}x$$

for $n = 1$, 2, 3, 4, and 5. [15]

Question 3

Oscar and Lucinda are planning to travel to Silverbridge by train and meet in the station café for lunch. They are starting from different cities, but both of their trains are due to arrive at Silverbridge at 1pm.

The driver of Oscar's train likes to observe herons. Herons may be taken to occur randomly and independently of each other subject to a certain average rate, such that the mean (expected) number that the driver will see during the journey is 1.5. Each time he sees a heron, there is a 10 minute delay as he stops the train and attempts to photograph the bird, although it has inevitably flown away before he can do so.

a) Find the probability that Oscar arrives at Silverbridge at 1.20pm. [2]

b) Give the mean and standard deviation of the number of minutes' delay in Oscar's train due to attempted heron observation. [3]

The driver of Lucinda's train is ill, so she travels by rail replacement bus. This introduces a delay which is a continuous random variable, equally likely to be any time between 0 and 30 minutes, but never less than 0 or more than 30.

c) Find the probability that Lucinda arrives at Silverbridge between 1.10pm and 1.15pm. [1]

d) Find the mean and standard deviation of the number of minutes' delay in Lucinda's journey. [3]

e) Find the probability that Oscar's train arrives before the mean (expected) arrival time of Lucinda's rail replacement bus. [3]

f) Find the probability that Lucinda's rail replacement bus arrives before the mean (expected) arrival time of Oscar's train. [2]

g) Find the probability that Oscar arrives at Silverbridge before Lucinda. [3]

h) Given that Oscar *does* arrive at Silverbridge before Lucinda, find the probability that the driver of his train saw more than one heron. [3]

Question 4*

a) Find

$$\lim_{x\to 0} \frac{x}{\tan x} \quad \text{and} \quad \lim_{x\to \frac{\pi}{2}} \frac{x}{\tan x} \quad . \text{ [2]}$$

b) Sketch a graph of $y = x\cot x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$. [3]

c) Define

$$I(\alpha) = \int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(\alpha \tan x)}{\tan x}\,\mathrm{d}x \quad .$$

Show that

$$I'(\alpha) = \alpha \int_0^{\infty} \frac{\mathrm{d}u}{(1+u^2)(\alpha^2+u^2)} \quad . \text{ [5]}$$

d) Find the integral in (c). [5]

e) Hence find $I(1)$, and so find

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cot x\,\mathrm{d}x \quad . \text{ [5]}$$