Post-Christmas IA NST Mathematics Test

January 2026

You have one hour. Complete any **two** questions from those printed below. The starred question requires knowledge of B-course material. The number of marks available for each part of a question is shown in square brackets. Begin your answer to each question on a new page, writing the question number clearly at the top. Do not submit answers to more than two questions. Calculators may not be used for this test.

Question 1

In this question z is a complex number which must always obey $|z+3i| = 5$.

a) Describe the locus of possible values of z. Sketch this locus on an Argand diagram, finding and marking on all intersections with the real and imaginary axes. [3]

b) Describe and sketch also the locus of $u = z - i$, marking on intersections as before. [2]

c) Describe and sketch also the locus of $v = iz$, marking on intersections as before. [2]

d) Find all values of z for which z^4 is a negative real number. (Leave your answers in exact surd form.) [6]

e) Sketch the locus of $w = z^2$, marking on all intersections with the real and imaginary axes as before. You may find the results of (d) helpful in finding some of these intersections. [5]

f) Consider the locus of $t = z^3$. State the largest and smallest values of $|t|$ and predict how many times the curve will cross itself. [2]

Question 2

There are two rings in three-dimensional space: an upper ring in the plane $z = b$ and a lower ring in the plane $z = -b$, where z is a Cartesian co-ordinate. Both rings are centred on the z-axis, and have radius a. A piece of string joins the upper ring at $\theta = \alpha$ to the lower ring at $\theta = -\alpha$, where θ is a cylindrical polar angular co-ordinate, and α is a fixed angle such that $0 \leq \alpha < \frac{\pi}{2}$.

a) Find a vector equation for the line of the string in the form $\mathbf{r} = \mathbf{p} + \lambda \mathbf{q}$, giving the vectors $\mathbf{p}$ and $\mathbf{q}$ in Cartesian form, and stating the range of λ. [3]

b) By taking the cross product of your equation from (a) with an appropriate vector, show that the points on the string all satisfy a vector equation of the form $\mathbf{r} \times \mathbf{s} = \mathbf{t}$, giving the vectors $\mathbf{s}$ and $\mathbf{t}$ in Cartesian form. [4]

c) Find the Cartesian co-ordinates of the point of intersection of the string with the plane $z = z_1$, where $-b < z_1 < b$. [2]

d) Find, by any method, the shortest distance between the string and the origin. [2]

e) Find the cylindrical polar co-ordinates of the point of intersection from part (c). [2]

f) Many more strings now connect the two rings. Each string takes a different value of a parameter β, and joins the upper ring at $\beta + \alpha$ to the lower ring at $\beta - \alpha$. The strings will all lie on a surface. Give the equation of this surface in cylindrical polar co-ordinates and in Cartesian co-ordinates, and sketch it. [6]

g) What happens to the surface in part (f) if $\alpha = \frac{\pi}{2}$? [1]

Question 3

The Museum of Probability carries out demonstrations of the 'Monty Hall Problem'. There are three large soundproof cupboards. Into one cupboard (chosen at random) is a (toy) car; in the other two cupboards are goats. (One of the 'goats' is a toy, but the other one is a real goat; the real goat is one of the Museum's principal attractions.) The Museum staff know which cupboard contains the car, but the visitors do not. A member of the Museum staff acts as host, and invites a visitor to select a cupboard. The host then opens one of the other cupboards, that the visitor had not selected, always doing so in such a way as to expose a goat (real or toy). The visitor is then allowed to change their selection for the other unopened cupboard if they wish. Finally, the visitor's chosen cupboard is opened, and they are said to have won if it contains the car, although the only prize they are actually allowed to take away from the museum is a commemorative postcard.

a) What is the probability that the visitor chooses the cupboard containing the car when making their initial selection? [1]

b) Given that the visitor does not change when allowed to do so, what is the probability that they 'win the car'? [1]

c) Given that the visitor *does* change when allowed to do so, what is the probability that they 'win the car'? [2]

d) Must the answers to (b) and (c) be the same? If so, why? If not, how can they be different? [2]

e) A particular visitor decides whether to switch randomly, by tossing a fair coin. What is the probability that they will 'win the car'? [1]

f) What is the probability that the visitor of part (e) did switch, given that they do 'win the car'? [3]

One day, the soundproofing in one of the three cupboards fails, so that the probability that the real goat is in a non-soundproof cupboard for a particular demonstration is now $\frac{1}{3}$. The real goat always bleats immediately after the visitor has made their initial selection, and not at any other time. If it bleats while in the non-soundproof cupboard, the visitor hears it and know which cupboard it is in.

g) Describe the best strategy for the visitor to use. (Note that when the visitor has selected the cupboard containing the car as their initial selection, the host will reveal a real goat or a toy goat with equal probability.) When using this strategy, what is the probability that the visitor will 'win the car'? [3]

h) Given that the visitor uses the best strategy and 'wins the car', what is the probability that the real goat's bleating was heard? [3]

i) On a certain day the demonstration is carried out 18 times. Find the mean and standard deviation of the number of times that bleating is heard. [4]

Question 4*

Schwarz's inequality is the result that

$$\left(\int_A^B f(x)g(x)\,\mathrm{d}x \right)^2 \leq \int_A^B (f(x))^2\,\mathrm{d}x \int_A^B (g(x))^2\,\mathrm{d}x$$

for any functions f and g.

a) Prove Schwarz's inequality. [8]

Define

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{sech}(x)}{1+x^2}\,\mathrm{d}x \quad .$$

b) Use Schwarz's inequality to find an upper bound on I. [5]

c) Use the inequality $(1+x^2)^{-1} \leq 1$ to find another upper bound on I. [4]

d) Use the inequality $\mathrm{sech}(x) \leq 1$ to find another upper bound on I. [2]

e) Determine which of (b), (c), and (d) gives the tightest bound. [1]