IA NST Maths Test for Jesus College

January 2013

You have one hour. Complete any **two** questions from those printed below. The number of marks available for each part of a question is shown in square brackets. Begin your answer to each question on a fresh sheet, writing your name and the question number clearly at the top. Calculators are forbidden.

Question 1

This question is about using complex numbers with a geometric construction to find an exact surd form for $\cos 72°$, and relating this to the golden ratio. Define $c = \cos 72°$, $s = \sin 72°$, and $z = e^{2\pi i/5}$.

a) Write a formula for z in terms of c and s. [1]

b) Write a relation between c and s that does not involve z. [1]

c) Draw an Argand diagram and mark on the positive integer powers of z from z to z^6 inclusive. Mark on the circle on which all of them lie. [3]

d) From your diagram, deduce a relation between the imaginary parts of z^2 and z^3. [2]

e) Using your answers to (a) and (d), together with (b), derive and solve an equation which is for c, and hence write c in an exact surd form. [8]

f) One definition of the golden ratio, ϕ, is that it is the length of the line joining z^2 to 1 divided by the length of the line joining z to 1. Write ϕ as the modulus of a complex number, giving the real and imaginary parts of this complex number in terms of c and s. [5]

Question 2

In this question a mole is a small mammal of the genus *Talpa*, and not a unit; the rood is a unit of area. Moles are distributed randomly, and independently of each other, in a certain area of pasture land, averaging λ animals per rood. An area is said to be infested if it has *more than* one mole per rood.

a) Field A is made by enclosing a single rood of the pasture land. Find the probability (in terms of λ) that field A (i) is not infested and (ii) is infested with moles. [2]

b) Field B is made by separately enclosing two adjacent roods of the pasture. Find the probability that neither half of field B is infested. [2]

c) Field C is made by enclosing a single two-rood area of pasture. Find the probability that field C, considered as a whole, is not infested. [3]

d) Which is more likely, (b) or (c)? Explain briefly in words how the difference arises. [1]

e) Farmers P and Q disagree on the answer to part (d), and make a bet: P will give Q £10 if neither half of field B is infested, and Q will give P £10 if field C, as a whole, is not infested. What is the expected gain of farmer Q? Sketch a graph of this expected gain as a function of λ, and find the value of λ which makes the modulus of the expected gain as large as possible. [7]

f) Suppose that $\lambda = 0.81$. What is the standard deviation of the number of moles to be found in field A? [2]

g) Suppose that $\lambda = 2$. Find the probability that fields A and C contain precisely two moles between them. [3]

Question 3

a) Find an equation for the plane P, which passes through the points $(-1, 1, 0)$, $(0, 2, 1)$, and $(1, 0, -1)$, (i) in vector form, and (ii) in Cartesian form. [5]

b) Surface S has vector equation $(\mathbf{r} - (17, 4, 1)) \cdot (\mathbf{r} - (17, 4, 1)) = 11$. Describe surface S completely. [3]

c) Consider the intersection of surface S with plane P. What shape is formed? Find the position of any relevant points in the plane, and any relevant lengths, to describe the shape fully within the plane. [8]

d) Find the co-ordinates of a point on the shape. [4]

Question 4

a) Factorize x^4+1 into two quadratic factors containing only real numbers. [6]

b) Hence, or otherwise, find

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4 + 1} \quad . \text{ [14]}$$