IA NST Post-Christmas College Maths Test

January 2015

You have one hour. Complete any **two** questions from those printed below. The starred question requires knowledge of B-course material. The number of marks available for each part of a question is shown in square brackets. Begin your answer to each question on a new piece of paper, writing your name and the question number clearly at the top. Fix together all of the pages of each answer into a bundle using the treasury tags provided. Do **not** fix the two bundles together. Calculators are forbidden.

Question 1

A dart is thrown onto a circular board in such a way that it is equally likely to land anywhere on the board. The random variable X gives the x-co-ordinate of the dart. The board has unit radius.

a) Find the probability density function for X, checking that it is normalized. [3]

b) Find the mean and standard deviation of X. [4]

c) Find the expected values of X^3, X^4, and X^5. [6]

d) Find the expected value of $\cos^{-1}|X|$. [4]

e) Find the expected value of $\frac{1}{1-X}$. [3]

Question 2

N darts are thrown onto a circular board in such a way that each dart is equally likely to land anywhere on the board.

a) Within the circular board is inscribed an n-sided regular polygon. The part of the board which is not within that polygon is painted blue. Find expressions for (i) the probability that an individual dart lies within the blue region, and (ii) the expected number of darts, B, in the blue region once all N have been thrown. [4]

b) A small circle in the centre of the board is painted red. This small circle has radius $1/n$ times the radius of the board. Find expressions for (i) the probability that an individual dart lies within the red region, and (ii) the expected number of darts, R_1, in the red region once all N have been thrown. [2]

Gamblers bet on the number of darts that will land in the red or blue regions. The gamblers want to know which region has the higher expected value, and by how much. They don't have a calculator, but they know that n is large. For the rest of the question, we shall assume that $\theta = 2\pi/n$ is a small quantity.

c) Find power series expansions for B and R_1 in terms of θ about $\theta = 0$ correct to the θ^2 term. Hence find the the first non-zero term in the power series expansion for $R_1 - B$, the expected excess of darts in the red region over the blue region, in terms of θ. Leave your answer in exact form, but state whether it is positive or negative, and hence whether R_1 or B is greater when n is large. [4]

d) The gamblers modify the game so that the red and blue regions have the same expected value to second order in θ, by multiplying the radius of the red region by a factor α, which does not depend on n. Find α. [3]

e) Due to higher-order terms in θ, the new expected value of the number of darts in the red region, R_2, is still not exactly equal to B. Find the first non-zero term in the power series expansion for $R_2 - B$ in terms of θ. [4]

f) The board is made with $n = 100$. Express the answer to (e) in terms of n and N. This is a good approximation to $R_2 - B$, as θ is small. Using the approximation that $\pi^2 \approx 10$, find the value of N required to make $R_2 - B$ equal to 1. [3]

Question 3

The function f provides a scalar value for every point in space. At the point with position vector $\mathbf{r}$, it is given by $f(\mathbf{r}) = (|\mathbf{r}| - \mathbf{r} \cdot \hat{\mathbf{g}})(|\mathbf{r}| + \mathbf{r} \cdot \hat{\mathbf{g}})$, where $\hat{\mathbf{g}}$ is a fixed unit vector.

a) Show algebraically that $f(\mathbf{r} + \lambda\hat{\mathbf{g}}) = f(\mathbf{r})$ for any λ. [5]

b) If $\mathbf{r} = \alpha\hat{\mathbf{g}} + \beta\hat{\mathbf{h}}$, where $\hat{\mathbf{h}}$ is a unit vector perpendicular to $\hat{\mathbf{g}}$, show that

$$\left|\sqrt{f(\mathbf{r} + \mu\hat{\mathbf{h}})} - \sqrt{f(\mathbf{r})}\right| = \mu$$

for any $\mu \geq 0$. [8]

c) Describe the surface $f(\mathbf{r}) = K$. [4]

d) Provide a geometrical interpretation of the function $f(\mathbf{r})$, and explain geometrically the results of (a) and (b). [3]

Question 4*

a) Prove Schwartz's inequality. [10]

b) Use Schwartz's inequality to find an upper bound for

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{1+x^2}\,\mathrm{d}x$$

expressing your answer as a fourth root of a multiple of a power of π. [10]