IA NST Post-Christmas College Maths Test

January 2016

You have one hour. Complete any **two** questions from those printed below. The starred question requires knowledge of B-course material. The number of marks available for each part of a question is shown in square brackets. Begin your answer to each question on a new piece of paper, writing your name and the question number clearly at the top. Fix together all of the pages of each answer into a bundle using the treasury tags provided. Do **not** fix the two bundles together. Calculators are forbidden.

Question 1

a) Find the Maclaurin series (i.e. the Taylor series about x=0) of the function

$$f(x) = \log_e\left(\frac{1}{1-x}\right)$$

writing explicitly the first four non-zero terms. [4]

b) Let

$$g(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n}$$

where x is not an integer multiple of 2π. By writing $g(x)$ as the real part of a complex function and using the result of part (a), show that

$$g(x) = -\log_e\left|2\sin\left(\frac{x}{2}\right)\right| \quad . \text{ [10]}$$

c) Sketch $g(x)$ for x between -2π and 2π, marking on the positions of all zeros and minima of the function. [6]

Question 2

Line L_1 has equation $2x + y = 5$ within the $z = 1$ plane. Point P_1 has $x = y = z = 3$. Plane Π_1 contains the line L_1 and the point P.

a) Find a vector equation for the line L_1 in terms of a free parameter λ. [3]

b) Find a vector equation for the plane Π_1 in dot-product form. [6]

c) Sphere S_1 has its centre at the point with position vector $5(\mathbf{i}+\mathbf{j})+3\mathbf{k}$, and is tangent to plane Π_1 at point P_T. Write an equation for sphere S_1 in the form $|\mathbf{r} - \mathbf{a}| = k$, where $\mathbf{a}$ and k are constants that you must find. [7]

d) Find the shortest distance between P_T and L_1. [2]

e) If the positive z-direction is upwards, is S_1 above or below the plane Π_1? How can you be sure? [2]

Question 3*

a) Show that

$$\int_0^\infty e^{-z^2}\,\mathrm{d}z = \frac{\sqrt{\pi}}{2} \quad . \text{ [6]}$$

b) The gamma function is defined as

$$\Gamma(x) = \int_0^\infty e^{-t}\,t^{x-1}\,\mathrm{d}t \quad .$$

By using a suitable substitution or otherwise, find $\Gamma(2.5)$, expressing your answer as an exact fraction times a power of π. [14]

Question 4

The Museum of Probability opens from 2pm until 5pm on Mondays, Tuesdays, Wednesdays, Fridays, and Saturdays. On Thursdays it opens from 2pm until 7pm. On Sundays it does not open at all. Staff training takes place on alternate Tuesdays from 11am to 1pm. The museum is not open during staff training.

Throughout the year 2015, the museum received visitors who arrived randomly and independently of each other, at an average rate of three per hour during the museum's opening times, independently of the time of day and day of the year. When in the Museum, visitors could select various counters from bags and biscuits from jars, as well as viewing exhibits related to probability.

a) What was the mean number of visitors arriving on a Monday afternoon in 2015? [1]

b) On a particular Wednesday afternoon in 2015, what was the probability that:

(i) no visitors arrived at all?

(ii) only three visitors arrived in total?

(iii) exactly three visitors arrived during each of the hours of opening?

(iv) there was one hour in which two visitors arrived, one in which three arrived, and one in which four arrived? (These hours can occur in any order.)

In each case give your answer as an exact fraction multiplied by a power of e. Arrange the answers to parts (i)-(iv) in descending numerical order. You may use the results that $9^3 = 729$ and $9^4 = 6561$. [14]

c) During the first few days of 2016, the heating in the Museum fails, due to a faulty gasket in the Museum's boiler. In addition, a new exhibit—a goat used in demonstrations of the Monty Hall problem—eats all of the chocolate biscuits. Staff suspect that visitor numbers will go down as a result, but the Director of the Museum believes that the visiting rate will remain the same as in 2015. In one particular week in 2016, there are only two visitors during the entire week. Show that the probability that this would happen given the 2015 visiting rate is less than 2×10^{-23}. [Hint: you may use the fact that $e^3 > 20$, and that $2^{10} = 1024 > 10^3$.] [5]