IA NST Post-Christmas College Maths Test

January 2017

You have one hour. Complete any **two** questions from those printed below. The starred question requires knowledge of B-course material. The number of marks available for each part of a question is shown in square brackets. Begin your answer to each question on a new piece of paper, writing your name and the question number clearly at the top. Fix together all of your answers into a bundle using the treasury tags provided, arranging the questions in numerical order. Calculators are forbidden.

Question 1

A regular octahedron has vertices $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$, $D(0, -1, 0)$, $E(-1, 0, 0)$, and $F(0, 0, -1)$.

a) Find the length of the line segment AB, which forms one edge of the octahedron. [1]

b) Use the dot product between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ to show explicitly that the angle between these two edges is $\pi/3$. [2]

c) The octahedron is placed such that face ABC lies on the ground. Find the heights of points D, E, and F above the ground. [4]

d) Find the vector area of the quadrilateral $BCDE$, and the area of its projection onto the ground. [4]

e) Find the volume of the octahedron. You may use the result that the volume of a tetrahedron $OXYZ$ is given by one-sixth of the triple scalar product of the vectors $\overrightarrow{OX}$, $\overrightarrow{OY}$, and $\overrightarrow{OZ}$. [4]

f) Find an equation for a cylinder that passes through all of the vertices of the octahedron, in the form $|\mathbf{r} - (\hat{\mathbf{p}} \cdot \mathbf{r})\hat{\mathbf{p}}| = k$, where k is a constant real number and $\hat{\mathbf{p}}$ a constant unit vector, giving the values of k and $\hat{\mathbf{p}}$. State the number of distinct possible choices for $\hat{\mathbf{p}}$. [5]

Question 2

a) Use Euler's formula to show that

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad . \text{ [1]}$$

b) Let $x = \cos y$. Using (a), solve the equation to show that

$$y = \frac{1}{i}\log_e\left(x + i\sqrt{1-x^2}\right) \qquad (*)$$

is one solution. [4]

c) Give a geometric interpretation of why equation $(*)$ works. Refer to an Argand diagram. [2]

d) Continuing your analysis from (b), find a formula for the other solution of $x = \cos y$ in the range $-\pi < y \leq \pi$, and show algebraically that it satisfies the expected symmetry of the cosine function. (Take $\log_e$ to give a principal value, with its imaginary part between $-\pi$ and π.) [4]

e) By differenting equation $(*)$ directly, find the derivative of the equation $y = \cos^{-1} x$ in its simplest form. [4]

f) Use equation $(*)$ to find a function f such that

$$2\cos^{-1} x = \cos^{-1}(f(x))$$

and explain why this particular function is to be expected. [5]

Question 3*

Let $y = x^k e^x$, where k is a positive integer.

a) Use the Leibniz rule to find $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$. Give separate answers for $n < k$ and $n \geq k$; give each answer in closed form using summation notation with the symbol Σ. [8]

b) Using your result from (a), find the value of $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ at $x = 0$ in the case $n \geq k$, giving your answer as a ratio of factorials. [4]

c) Use your result from (b) to find the Maclaurin expansion for y in terms of x. Show that it is the product of the Maclaurin expansions for x^k and e^k, as expected. [4]

d) Apply an appropriate convergence test to show that the series in (c) converges for all real values of x. [4]

Question 4

The Museum of Probability employs $n+1$ staff. One of them is the Director; the other n are ordinary workers. All of them currently suffer from a disease meaning that, on each morning, they are unfit for work that day with a certain probability, u. This is independent of their fitness for work on other days, and of their colleagues' fitness for work.

For the Director, $u = P$. For each ordinary worker, $u = p$. The museum also has a goat, which is used in demonstrations of the Monty Hall problem. The goat is always fit for work, because the disease does not affect goats; however, it does not count as a member of staff because it would be unable to supervise evacuation of the Museum in case of emergency. For the Museum to be allowed to open, there must be at least two staff who are fit to work.

In parts (a) to (d) we are considering one, randomly chosen day, and you should give your answers as formulae in terms of P, p, and n.

a) Find the probability that all of the staff are unfit for work. [1]

b) Find the probability that the Director, and all but one of the ordinary workers, are unfit for work. [3]

c) Find the mean number of staff who are unfit for work. [1]

d) Find c, the probability that the Museum has to close due to staff illness. [3]

e) Show that, given that the Museum is closed due to staff illness on 4th January, the probability that the Director is unfit for work on that day is

$$P\left(\frac{p+n(1-p)}{p+nP(1-p)}\right) \quad . [3]$$

f) Suppose that p is sufficiently large that $n(1-p) \ll p$. Find an approximation to your result in (e) in this case, and explain why this approximation makes sense in this limit. [2]

g) Suppose that p is sufficiently small that $p \ll nP(1-p)$. Find an approximation to your result in (e) in this case, and explain why this approximation makes sense in this limit. [2]

h) The Museum should be open for the six days from 9th to 14th January inclusive. Find in terms of c the probability that it will have to close due to staff illness for at least two successive days during this period. Expand your formula to third order in the small quantity c. [5]