IA NST Post-Christmas College Maths Test

January 2018

You have one hour. Complete any **two** questions from those printed below. The starred question requires knowledge of B-course material. The number of marks available for each part of a question is shown in square brackets. Begin your answer to each question on a new piece of paper, writing your name and the question number clearly at the top. Do not submit answers to more than two questions. Calculators are forbidden.

Question 1

Throughout this question the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{r}$ are vectors in three dimensions; their components are real numbers.

a) Give a specific example of values of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ such that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \quad . \quad [2]$$

b) Give a specific example of values of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ such that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \quad . \quad [2]$$

Condition E is the condition that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{r}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{r}$$

where $\mathbf{a}$ and $\mathbf{b}$ are fixed and we consider different values of $\mathbf{r}$.

c) Let $\mathbf{a} = (-101, 102, 103)$ and $\mathbf{b} = (-104, 105, 106)$. Describe fully the locus of values of $\mathbf{r}$ that satisfy condition E. [5]

d) Let $\mathbf{a} = (-3, 4, 2)$ and $\mathbf{b} = (4, 5, -4)$. Describe fully the locus of values of $\mathbf{r}$ that satisfy condition E. [5]

e) Show that there can be no values of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ that satisfy the equations

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{0}. \quad [6]$$

(Hint: first show that the three vectors would have to be coplanar, and then that this leads to a contradiction.)

Question 2

The Museum of Probability will be closed throughout March in order to save on staffing costs. The Director will go on holiday to Barbados. The other staff, who are on zero-hours contracts, will be advised to take other jobs in the "gig economy" if necessary to make ends meet. The museum's goat, which is used in demonstrations of the Monty Hall problem, will be lent to a petting zoo. Only the museum's most junior employee, the 19-year-old trainee, will come in to the museum on certain days, in order to receive deliveries of parcels sent to the museum. The trainee is paid £6.40 per hour.

On a certain day in March, the trainee has to receive one parcel, which a logistics company will deliver at a random time between 11am and 5pm, such that any time is equally likely. Let t be the time elapsed between 11am and the delivery of the parcel.

a) Sketch the probability density function for t. [2]

b) Sketch the cumulative distribution function for t. [2]

c) Find (or state) the mean of t. [1]

d) Find the standard deviation of t. [3]

On another day later in March, the trainee has to receive n parcels, each delivered by a different logistics company. Each parcel will arrive at a random time between 11am and 5pm, independently of the other parcels. The trainee will arrive at 11am and leave work once the last parcel has been delivered; he does not have to wait until 5pm. Let u be the time which he spends at work that day.

e) Find and sketch the cumulative distribution for u. [2]

f) Find and sketch the probability density function for u. [2]

g) Find the mean value of u. (Note that this will depend on n). [3]

h) On a particular day, $n = 3$. The Director wants to sort out the museum's wage bill in advance, and decides to pay the trainee £28.80 for the day, because this is the trainee's standard hourly rate multiplied by the expected value of u as found in part (g). The trainee agrees in principle, but explains that he would like to reduce the variance in the effective hourly rate he will receive by making a bet with the Director. He suggests that, if the last parcel arrives after 3.30pm, the Director will pay him £6.40, but if the last parcel arrives before 3.30pm, he will pay the Director £7.04 (i.e. 10% more than the £6.40). The trainee suggests that the Director should take the bet because the possible gain (for the Director) is slightly greater than the possible loss. The Director agrees. Find the expected value of the Director's gain from the bet, as an exact number of pence. [5]

Question 3*

Let

$$I = \int_0^{2\pi} \cos^{2n} x \, \mathrm{d}x \quad \text{and} \quad J = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n} x \, \mathrm{d}x$$

where n is a positive integer, possibly large.

a) By means of a diagram show that $I = 2J$. [2]

b) Find (by any method) the first non-zero term in the Maclaurin expansion of $\log_e(\cos x)$. [3]

c) By using $\cos^{2n} x = e^{2n \log_e(\cos x)}$ with part (b) and the standard result (which you may quote) for $\int_{-\infty}^{\infty} e^{-\lambda z^2} \mathrm{d}z$, show that $J \approx \sqrt{\pi/n}$ for large n. [5]

d) By writing $\cos x$ in terms of e^{ix} and e^{-ix}, show that

$$I = \frac{2\pi(2n)!}{2^{2n}(n!)^2} \quad . \quad [6]$$

e) Stirling's approximation (which works for large n) is sometimes given as $n! \approx n^n e^{-n}$, (which was derived in lectures) but a more accurate form is $n! \approx k n^\alpha n^n e^{-n}$, where k and α are constants which we can find. Using the results of (a), (c), and (d) together, show that they are consistent with this latter form of Stirling's approximation, and find the values of k and α required to make it work. [4]

Question 4

a) Sketch on the same axes graphs for $x > 0$ of

$$f(x) = \frac{1}{\sqrt{1+x^2}} \quad \text{and} \quad g(x) = \frac{1}{1+x} \quad . \quad [4]$$

b) Find exact formulae for $\int_0^a f(x) \, \mathrm{d}x$ and $\int_0^a g(x) \, \mathrm{d}x$. [4]

c) By expanding $f(x)$ as far as terms in x^4 and integrating, or otherwise, find a series expansion for $\int_0^a f(x) \, \mathrm{d}x$ valid for small a, correct up to and including the term in a^5. [5]

d) Now consider the behaviour of these integrals when a is large. Combine your results from part (b) to find an exact formula for $\int_0^a (f(x) - g(x)) \, \mathrm{d}x$ as a single logarithm. Write this in terms only of a^{-1} and find a series expansion in ascending powers of a^{-1} correct to terms in a^{-2}. [5]

e) State which of the integrals $\int_0^\infty f(x) \, \mathrm{d}x$, $\int_0^\infty g(x) \, \mathrm{d}x$, and $\int_0^\infty (f(x) - g(x)) \, \mathrm{d}x$ exist, giving the value of any that do exist. [2]