IA NST Post-Christmas College Maths Test

January 2020

You have one hour. Complete any **two** questions from those printed below. The starred question requires knowledge of B-course material. The number of marks available for each part of a question is shown in square brackets. Begin your answer to each question on a new piece of paper, writing your name and the question number clearly at the top. Do not submit answers to more than two questions. Calculators may not be used for this test.

Question 1

Sketch the locus of z on an Argand diagram for each of the following equations.

a) $\mathrm{Re}(z) + \mathrm{Im}(z) = 1$ [1]

b) $|z + 1 - i| = 1$ [2]

c) $\dfrac{1}{\sqrt{2}} = \left|1 - \dfrac{1}{2z} + \dfrac{i}{2z}\right|$ [6]

d) $\arg(z - 1) = \dfrac{\pi}{4}$ [2]

e) $\mathrm{Re}\left(\dfrac{1}{i}\log_e\left(\dfrac{z-1}{z-i}\right)\right) = \dfrac{\pi}{4}$ [6]

f) $|\log_e z|^2 - (\log_e |z|)^2 - \dfrac{\pi}{2}\arg(z) + \dfrac{\pi^2}{18} = 0$ [3]

Question 2

The vectors $\mathbf{r}$ and $\mathbf{s}$ obey

$$\mathbf{a} \times \mathbf{r} + (\mathbf{a} \cdot \mathbf{r})\mathbf{b} = \mathbf{c} \quad \text{and}$$

$$\mathbf{a} \times \mathbf{s} + (\mathbf{b} \cdot \mathbf{s})\mathbf{a} = \mathbf{c}$$

where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are constant non-zero vectors, none of which is perpendicular to any of the others.

a) Find a formula for $\mathbf{r}$. [9]

b) Find a formula for $\mathbf{s}$. [11]

Question 3*

Define

$$C_n(x) = \frac{\mathrm{d}^n(\cos(x^2))}{\mathrm{d}x^n}$$

and

$$S_n(x) = \frac{\mathrm{d}^n(\sin(x^2))}{\mathrm{d}x^n}$$

where x is a real number.

a) Use the Leibnitz formula to find an expression for $S_n''(x)$ in terms of $C_n(x)$ and $C_n'(x)$. [8]

b) Find a similar expression swapping the rôles of S_n and C_n. [2]

c) Hence, or otherwise, show that

$$\frac{\mathrm{d}^4 C_n(x)}{\mathrm{d}x^4} + 4x^2 \frac{\mathrm{d}^2 C_n(x)}{\mathrm{d}x^2} + 4(2n+5)x \frac{\mathrm{d}C_n(x)}{\mathrm{d}x} + 4(n+1)(n+3)C_n(x) = 0 \quad . \quad [10]$$

Question 4

The Director of the Museum of Probability has introduced a café into the Museum. He hopes to entice more visitors in with coffee and cake.

The Director periodically buys a large cake from a local supplier. The quality, Q, of each cake decays exponentially with time, starting from 1. When $1 \geq Q \geq e^{-1}$, the Museum's visitors rate the cake as "acceptable". When $Q < e^{-1}$ the Museum's visitors rate the cake as "stale". Only when Q falls to Q_0, where $Q_0 < e^{-1}$, however, does the Director declare the cake to be unfit for human consumption. There are always some slices remaining at this point, and the Director feeds them to the Museum's goat, which is used for practical demonstrations of the Monty Hall problem. The goat always rates the cake as "delicious", but its opinions do not count because it is unable to express them on social media.

The probability density function for the value of Q of a slice of cake ordered at a random time in the Museum's café is

$$f(Q) = \frac{k}{Q} \quad \text{for} \quad Q_0 < Q < 1$$

where k is a constant.

a) Find k and sketch the probability density function $f(Q)$. [3]

b) Find and sketch the cumulative distribution function for Q. [3]

c) Find the mean (expected) value of Q. [2]

d) Find the median value of Q. (This is the value of Q that a random slice of cake is equally likely to be above or below.) [1]

e) Find the modal (most probable) value of Q. [1]

For the rest of the question, take $Q_0 = e^{-6}$.

f) Find numerical values of the mean, median, and modal values of Q correct to two significant figures, by using the approximation $e^3 \approx 20$. [3]

g) Find the probability that a random visitor eating cake in the café will rate it as "stale", expressing your answer as an exact fraction. [1]

h) Three friends visit the Museum's café independently of each other. Each tries a slice of cake. Find the probability that no more than one of them rates a cake as "stale", expressing your answer as an exact fraction. [3]

The Museum of Intellectual Art also opens a café. It buys its cakes from the same supplier as the Museum of Probability, so their value of Q decays in the same way. However, it gives its cakes away to starving artists as soon as they become stale, so that visitors to its café always rate the cake as "acceptable".

i) Oscar and Lucinda toss a fair coin. If it shows heads, they visit the Museum of Probability. If it shows tails, they visit the Museum of Intellectual Art. When in the museum that they are visiting, they share a slice of cake, and find that it is "acceptable". What is the probability that they are in the Museum of Probability? [3]